

## Introduction to Measurement Uncertainty and Error Analysis

Welcome to university physics.

What you'll discover in your courses is that science builds up a "model" describing how the world and the universe work. If we understand the model then we can make predictions as to how things will behave, and we can design complex machines and systems. If we're clever, our understanding of science may have meaningful impact on the health and welfare of the people and the planet around us. Knowledge is important and powerful.

The foundation of this picture of how things are related stems from observation, theory, and experimentation: the *scientific method*. When we observe what we perceive to be a relationship between two properties (or variables) of a system we might hypothesize that the relationship takes on a particular mathematical form. Ideally we can then take measurements, controlling one variable while watching the second. If our observations fit our theory then we've succeeded in coming to some form of deep understanding. If the results don't match our expectations, then our theory needs to be modified and improved. By this process of refinement we improve our understanding. Our experiments are a critical part of the scientific method, and as such, we need to ensure we go about performing and reporting them in ways that are thoughtful, repeatable and verifiable, and convincing to others. The laboratory component of a science program serves this important purpose.

### 1. Comparisons to Theory or Accepted Values - Percentage Error

Before we step into the minutia of measurements and calculations, we should take a moment to state an important reality of experimental physics: measurements and calculations, in and of themselves, are *rarely* interesting.

What *is* interesting though is when, by virtue of our measurements and calculations, we can compare a result to an accepted value or validate a theoretically predicted value.

By way of example, if we have devised an experiment and measured the acceleration of a falling object to be  $9.78 \pm 0.05 \text{ m/s}^2$ , this is a fine outcome. We have made our measurement, and we have conscientiously quantified the uncertainties in our measurements such that we're confident our result is correct to  $\pm 0.05 \text{ m/s}^2$ , possibly due to instrumentation limitations or slight variations in our results from one trial run to the next. However, our measurement lacks any meaningful context by itself. Our result might apply to one object only, and for all we know, other objects could fall at completely different acceleration rates.

If however we consider Newton's law of gravitation, we realize that at the surface of the earth we expect that

$$g = -GM/r^2 = 9.80665 \text{ m/s}^2,$$

since  $G$  (the gravitational constant),  $M$  (the mass of Earth), and  $r$  (the radius of Earth) are already known to quite good accuracy and precision.<sup>1</sup> We can report (happily) that our result, within the limitations of our experimental uncertainty, is "consistent" with the accepted value of  $g$ . A calculation of "Percentage Error," where

---

<sup>1</sup> Precision refers to the number of significant figures in a measurement. In general, the better the resolution of our tools or the more samples we can measure, the better our precision. Accuracy meanwhile refers to how "realistic" our measurements are. Systematic errors like poorly calibrated instruments mean that we might achieve results with good precision but poor accuracy.

$$\%Error = \frac{|Measured - Accepted|}{Accepted} \times 100\%$$

can be useful to convey the accuracy of our result, showing how closely it matches accepted values or theoretical predictions.

Additionally, our uncertainty ( $\pm 0.05\text{m/s}^2$ ) is deemed “insignificant” since, within the range of our uncertainty, our result overlaps the accepted value. Given our apparently successful result, we might therefore extend our study to objects of other masses, or go further and study objects subject to additional forces, like air resistance.

If, on the other hand, we had measured the acceleration due to gravity to be  $7.5 \pm 0.7\text{m/s}^2$  then we must report that our result is “inconsistent” with the commonly accepted value of the acceleration due to gravity.

In this second case, the discrepancy between our result and the accepted value in our gravity example is “significant” because our result is quite different than the expected value, even at the extremes of our range in uncertainty. The percentage error when compared against the accepted value is approximately 23%. This situation would probably lead us to re-evaluate our calculations (were mistakes made?), our methodology (does our experiment have flaws we overlooked?), and our tools (are systematic problems affecting our results, e.g. slow/fast timers, or poorly calibrated length scales?). We may need to rethink our experiment. Or perhaps we find ourselves in the interesting situation that our experiment is correct and we have discovered a flaw in the theory we’re investigating (many unexplained observations and results have eventually led to historic scientific revolutions!).

Finally we should consider a case where our experiment tells us that acceleration due to gravity is, say,  $11 \pm 2\text{m/s}^2$ . In this case our result is “consistent” with the accepted value of  $9.80665\text{m/s}^2$ , however our percentage error ( $\sim 12\%$ ) and uncertainty ( $\pm 2\text{m/s}^2$ ) are both relatively large. In some scenarios, depending on the experimental conditions, this might be a completely appropriate and acceptable outcome, but it might also indicate that improvements in methodology or instrumentation are required to better our precision and confirm the accuracy of our result. We would then need to consider whether efforts at improving our experiment would be warranted and/or successful.

## 2. Commonalities Between Experiments

While most of our experiments will touch on a variety of unique and apparently unrelated topics, they will generally have in common that we will compare a result to an accepted value or validate a result predicted by theory, as in the example scenarios above. This serves as an effective way to learn to experiment, and eventually students may find themselves involved in research investigating entirely unknown and new science.

## 3. Measurements and Uncertainty

Let us now begin to formalize our approach to dealing with experimental uncertainties,<sup>2</sup> mathematically speaking. We have three important tasks ahead:

---

<sup>2</sup> Note, in an experimental context, the term “uncertainty” is often used interchangeably with the term “error.” Neither term refers to an error on the part of the researcher, but rather to the quantifiable limitations of the experimental tools or methods. We will usually use the term “uncertainty” in this document, but you might encounter both in your studies.

- 1) We must establish how uncertainty arises in experiments. We'll see that in most experiments there are various factors contributing to uncertainty, and depending on the scenario, some may be significant while others might be negligible.
- 2) We must learn to quantify uncertainty in our experiments. To some extent this will require common sense, both to minimize uncertainties where possible and to assign them "reasonable" values. We'll address the three main types of uncertainties as we proceed.
- 3) We must finally determine how to correctly carry, or "propagate," these uncertainties through our calculations. Our experiments will often require directly measuring two (or more) properties of a system in order to calculate a third dependent property (e.g. by hanging a known mass from a spring,  $k=F/x$  we can "indirectly" determine the spring's constant,  $k$ , if we can "directly" measure the displacement,  $x$ , caused by the known force,  $F=mg$ ). We will need to know how the uncertainties in the measured values carry through to affect the final result.

### **Notation**

In your classes you should notice that you encounter examples almost always dealing in numbers that are exactly defined. As such, the solution to a problem usually has a single "correct" answer:  $a+b=c$ , or  $xy=z$ . This is a convenient way to teach theory and to practice calculations. However, this tendency to deal in exact numbers risks that students develop an unrealistically "clean" perception of the science they're studying.

In the practical reality of experimental physics we must deal with the fact that measurements are never known with absolute certainty. No matter the effort applied, even the best experimentalist must contend with unavoidable limitations in how confidently they can express a measurement.

Therefore when we measure a mass, we require a notation to capture both our measurement and also this "uncertainty" in our measurement. In real terms we must express our result not as an exact value " $m$ ," but rather as a best effort at measurement of  $m$ , with an uncertainty recorded to indicate our confidence in our measurement. That is, the preferred form for recording  $m$  would be

$$m = m_{(best\ measurement)} \pm \Delta m,$$

or, even more succinctly, we simply record the mass as  $m \pm \Delta m$ . Similarly, lengths would be expressed as  $L \pm \Delta L$ , times expressed as  $t \pm \Delta t$ , momentum as  $p \pm \Delta p$ , etc.

This serves as a useful notation when dealing with individual measurements and also with larger sets of data, and in all labs this will be our preferred way of recording any measurement.

## **4. Sources of Uncertainty**

Uncertainty in experiments arises from three basic sources: reading uncertainties, random uncertainties, and systematic uncertainties. As a scientist, our job is to recognize where each type enters into an experiment, and to minimize their influence to the best of our ability. Ideally we can make two out of the three negligibly small compared against their more influential counterpart (that is, we identify which of the three sources of uncertainty most affects our measurement). We then simply quantify the source of uncertainty and propagate it through any calculations to arrive at our final result.

### **4.1 Reading Uncertainty**

Frequently we encounter situations where our instrumentation is a limiting factor in our ability to make a measurement.

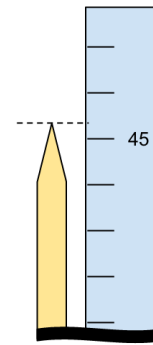
Because most instruments have an incremented scale, when a measurement falls between the smallest marked increments (“ticks”) it’s often appropriate and acceptable to estimate, or “interpolate,” this final digit.

For example, if a ruler has millimeter markings, the length of an object might be observed to be slightly greater than, say, 45mm but definitely less than 46mm. In this case we can look closely and, by our best effort, we decide that the object’s length is very close to 45.4mm.

However, we’re also cognizant that because we’re essentially estimating our final digit we may be slightly in error, and so we include in our recorded measurement an estimate of our uncertainty.

By common convention, we often choose to use half the value of the smallest increment on our measuring instrument as our estimate of the uncertainty. So in the case of our ruler measurement with millimetre markings, we would record our result as  $45.4 \pm 0.5 \text{mm}$ .

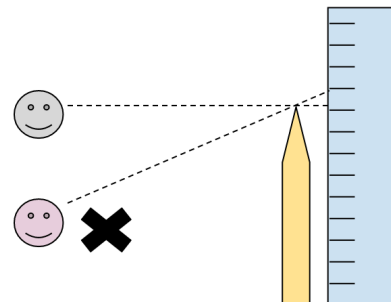
Note though while the “half the smallest division” convention is generally a good rule of thumb, we may still encounter cases where it would overestimate our uncertainty significantly. This tends to occur if our instrument’s divisions are widely spaced. If a ruler provided only centimeter divisions for instance, then we might quite reasonably estimate our measurement to an uncertainty of  $\pm 0.1 \text{cm}$  rather than the more conventional  $\pm 0.5 \text{cm}$ . To use  $\pm 0.5 \text{cm}$  simply by convention would in this case be overly imprecise, which would affect any subsequent calculations. We always need to apply good judgment in order to achieve best results.



**Figure 1.**  
**Measurement interpolation.**

### ***Parallax***

When reading measuring instruments with scale markings we must also be careful to read from directly in front of the instrument’s scale in order to avoid “parallax.” Parallax can occur if the experimenter moves their head to either side of the plane perpendicular to the scale and the object. This can cause a visual angular misalignment between the object and the measuring scale, such that the experimenter misreads or mis-interpolates their measurement.



**Figure 2. Parallax adds reading error.**  
**Measuring tools should be viewed “straight on.”**

### ***Digital Displays***

Electronic measuring instruments frequently come equipped with digital readouts or will display results to a computer. As such, we need to be able to record our results correctly when using these tools.

One needs to be somewhat careful in deciding on the appropriate uncertainty to record when using a digital display. For example, if an ammeter display reads, say, 10.51mA then it could be reasonable to believe the worst case uncertainty would be plus or minus the smallest digit, i.e.  $10.51 \pm 0.01 \text{mA}$ . However, this may not always be the case. Depending on the equipment, a more representative recording of the measurement might actually be  $10.510 \pm 0.005 \text{mA}$  (you would need to check the instrument labeling or manual to confirm the applicable convention). And with some instrumentation, you might find a label indicating that the reading is “accurate to  $\pm 1\%$  of full scale,” (such that in the case of our ammeter, if the full scale covers 20mA, then our uncertainty would be 1% of 20mA, or 0.2mA, leading to a reading of  $10.5 \pm 0.2 \text{mA}$ ). Despite these variations in reading, with

a little effort you'll be able to identify the most appropriate way to record your measurement and its uncertainty.

Finally we mention that in some cases the reading from a digital display may be "noisy" and fluctuate to some degree. (This can and does occur for analog displays as well, as when a needle on the display jumps about erratically). This can be a result of random variations influencing the experiment or the measuring instrument itself, such as electrical noise, thermal fluctuations, etc. We'll see next that random fluctuations, if non-negligible, may require us to increase our estimates of uncertainty to values larger than those justified by "reading uncertainty" alone (that is, our "half the smallest division" convention or " $\pm 1$  digit" on a digital display may significantly underestimate the realistic value of our uncertainty). We'll also see that for such cases, if we can repeat the measurement several times, statistical analysis will provide a more methodical mathematical way of determining our measurement and uncertainty.

## 4.2 Random Uncertainty

While "reading uncertainty" is applicable and useful when the object or property we are measuring is reasonably well defined, in many cases the "thing" we are measuring is naturally "fuzzy." In another sense, if we have difficulty defining when or where our measurement should start, repeating the measurement may give slightly varying results; there is an imprecision in our technique that is difficult to eliminate, so despite best efforts, some of our measurements are slightly too large or too small. In such cases the instrument reading uncertainty may become negligible compared to larger "random" fluctuations in the object or property being measured. As such, where random uncertainties dominate the measurement we must determine how to assign a more appropriate best estimate of uncertainty.

There are a few cases where such random uncertainties become dominant in a measurement.

### *Problem of definition*

Consider the difficulty in using a lens to focus a sharp image of an object onto a screen. According to the "thin lens equation" we expect that

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f}$$

where  $d$  is the distance between object and lens,  $d'$  is the distance between image and lens, and  $f$  is the lens focal length. If  $d$  is large,  $1/d \sim 0$  and therefore  $f \sim d'$  so the image location should equal the lens's focal length. However in practice, the image formed by the lens may look focused and sharp as the position of the screen is adjusted over many millimetres.

This can be a case where the image distance is inherently difficult to define, so we encounter an unavoidable randomness in our selection of where we believe we observe the sharpest image location. As such, even though we might measure the distance between lens and image using a ruler with millimeter markings, a larger uncertainty of  $\pm 0.5$ cm or even more might be the appropriate and more representative value. (You can see the difficulty for yourself by using a magnifying lens to focus the image from your window onto a paper screen in a darkened room.)

Similarly, consider the case where we are trying to measure the period of a pendulum. Using a stopwatch we count out the time required for ten oscillations, and then divide accordingly. We might discover that the time required for ten oscillations is 12.62s and conclude that the period of a single oscillation is  $1.262 \pm 0.005$ s, and we might be quite happy with our result. However, if we then repeat the measurement perhaps we would find that in our second attempt, ten oscillations take 12.28 s, so

that one oscillation takes  $1.228 \pm 0.005$  s. Of course we notice the “significant” discrepancy between our results (they’re quite close, but their uncertainties don’t overlap) and become concerned.

On closer consideration we probably realize that our ability to start the stopwatch at exactly the beginning of an oscillation and to stop at exactly the end of an oscillation are somewhat error-prone. There is in fact a degree of randomness that affects how well we start and stop our timer. We repeat the measurement several more times and observe periods of 1.262, 1.228, 1.211, and 1.254 s. We realize we can average these readings such that our most likely oscillation period is

$$T = [1.262 + 1.228 + 1.211 + 1.254] / 4 = 1.239 \text{ s}$$

and we could reasonably estimate the uncertainty to be  $\pm 0.003$  s based on the range we see in our value, giving a best measurement of  $T = 1.24 \pm 0.03$  s per oscillation. (Keep in mind that the stopwatch may not represent the only significant source of error in such an experiment, and we should remain mindful and observant lest we overlook another. Perhaps the way in which we set the pendulum into oscillation, its amplitude, or even air friction, influence the result. The possibilities are broad and very experiment-dependent.)

### Statistical Determination of Uncertainty

In such experiments or studies where there are many samples available to measure, or where the same measurement can be made many times, statistical analysis allows us a more mathematical approach for calculating our best measurement and uncertainty value than the interpolation or best estimates we’ve opted to use thus far.

Luckily, in many cases we’ll encounter in physics, we can indeed repeat our measurements. If we can take the same measurement several times, random uncertainties will cause some of our measurements to be slightly too large or slightly too small. The set of measurements will exhibit a “distribution” about the most probable “best value.” As a result, when averaged, we expect the random uncertainties to cancel out, providing a very good and representative final result.

We can calculate the average, or “mean” value,  $\bar{x}$ , as follows:

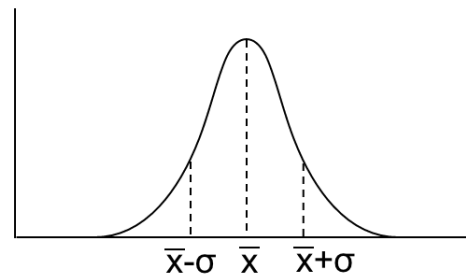
$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum_i^n x_i}{n}$$

We can gain a measure of the average variance of our data from the mean by calculating the “standard deviation,”  $\sigma$ , according to:

$$\sigma = \sqrt{\frac{\sum_i^n (x_i - \bar{x})^2}{(n-1)}}$$

This standard deviation indicates the average separation of a data point from the mean. Note that as the number of data points recorded increases, this average value should remain relatively constant.

Finally, as our measure of uncertainty in the mean value of our data set, we calculate the “standard deviation of the mean,”  $\sigma_m$ , given by:



**Figure 3: Probability distribution for a set of measurements subject to random uncertainties, showing mean value and standard deviation.**

$$\sigma_m = \frac{\sigma}{\sqrt{n}}.$$

Here we note that as we increase the number of measurements recorded,  $n$ , the standard deviation of the mean decreases (thus the precision with which we know  $\bar{x}$  increases). So by taking more measurements, we are able to increase our overall precision. Note though that while useful to a point, the technique can become time consuming and yields diminishing returns, as for instance, to increase  $\sigma_m$  by a factor of 10 we must increase the number of measurements by 100.

So, in summary, given a set of data from a repeatable measurement subject to random variations, our best result is given by:

$$x \pm \Delta x = \bar{x} \pm \sigma_m = \bar{x} \pm \frac{\sigma}{\sqrt{n}}.$$

Given the option, we should always repeat a measurement several times and use the mean and standard deviation of the mean as our values of best measurement and uncertainty, respectively. This provides the most rigorous method available to confidently and precisely determine our result, whereas when forced to interpolate or estimate the risk of significantly over- or under-estimating our measurement is much greater.

### 4.3 Systematic Uncertainty

Systematic uncertainties are, unfortunately, often the most difficult to identify and account for in an experiment.

Typical examples would be calibration errors in our instruments: for instance, a clock that runs too fast, or a ruler whose markings are stretched or scaled incorrectly. To acquire a set of measurements using a stopwatch that consistently runs too slowly for instance, we would of course still encounter the random fluctuations that govern the precision of our estimate (or statistical calculation) of uncertainty. But rather than acquiring a set of data distributed about the true value, all measurements would skew slightly to smaller readings than should properly be the case. The result might be a precise looking result, but one that is ultimately inaccurate.

Similarly, subtle effects like air pressure or temperature changes in the laboratory might have significant influence on the outcome of measurements, unbeknownst to the experimenter.

Even experimenter bias might contribute to introduce systematic uncertainty. Think of parallax, described earlier as the tendency to misread an instrument's scale if viewed from an angle. Normally the experimenter would take readings from directly in front of the scale, but could be forgiven for introducing small random variations if the head or eye position shifts slightly left or right (with relatively equal tendency). To consistently read the scale from an angle always left of straight on would tend to skew the readings. This bias, or ones like it, should be identified and corrected.

Overall, it can be quite difficult to compensate for systematic uncertainties. Better-calibrated tools (or recalibrating the tools available against certified calibration standards) can improve results, as can attention to details, like parallax, that might be surprisingly influential.

## 5. Propagation of Uncertainty

As you can see, recording a good and reliable measurement, though not necessarily difficult, does require awareness of where one risks over- or under-estimating uncertainty. Interpolating or

estimating random variations are sometimes necessary, but if done badly can significantly misrepresent the actual uncertainty, giving a result with unrealistic precision. Statistical analysis provides the most methodical and mathematically rigorous approach available for identifying a reliable and precise mean value (best measurement) and standard deviation of the mean (uncertainty). However, systematic uncertainties, if not identified and minimized, could still render a highly precise effort at a measurement completely inaccurate. Now that we're aware of these possible pitfalls though, we can endeavor to be both precise and accurate, which is worthwhile if we want our experiments to be successful.

So what we have achieved thus far is the knowledge needed to acquire a set of good measurements. We should be able to minimize or eliminate avoidable sources of error, and we can therefore record our results with realistic uncertainties.

### 5.1 Uncertainty Propagation for Addition and Subtraction

Often in experiments we'll be taking direct measurements and performing calculations to determine (indirectly) a property of interest. Whereas in our "clean" textbook examples,  $a+b=c$  or  $xy=z$ , we must now develop a formalism for handling the more realistic calculations where, for instance,

$$(a \pm \Delta a) + (b \pm \Delta b) = (c \pm \Delta c) \quad \text{and} \quad (x \pm \Delta x)(y \pm \Delta y) = (z \pm \Delta z).$$

We begin by deriving  $\Delta c$  for the case of addition of two numbers that include uncertainties. If we consider the extremes  $a+\Delta a$  and  $b+\Delta b$ , then we can regroup as follows:

$$\begin{aligned} (a + \Delta a) + (b + \Delta b) &= (a + b) + (\Delta a + \Delta b) \\ &= c + \Delta c \end{aligned}$$

Similar consideration applies to the opposite extreme where we use  $a-\Delta a$  and  $b-\Delta b$ .

$$\begin{aligned} (a - \Delta a) + (b - \Delta b) &= (a + b) - (\Delta a + \Delta b) \\ &= c - \Delta c \end{aligned}$$

Thus, in the case of addition,

$$\begin{aligned} (a \pm \Delta a) + (b \pm \Delta b) &= (a + b) \pm (\Delta a + \Delta b) \\ &= (c \pm \Delta c) \end{aligned}$$

and we see clearly that our uncertainties can be expressed as

$$\Delta c \sim \Delta a + \Delta b.$$

An identical result can be derived for subtraction, where

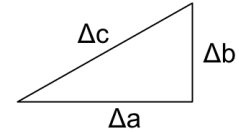
$$(a \pm \Delta a) - (b \pm \Delta b) = (c \pm \Delta c).$$

We should note that this expression "slightly" overestimates  $\Delta c$  because statistically our average uncertainties  $\Delta a$  and  $\Delta b$  will be somewhat less than the limiting value we used in our illustration. We instead use the expression

$$\Delta c = \sqrt{\Delta a^2 + \Delta b^2}.$$



We can see that the expression is mathematically equivalent to the Pythagorean theorem governing the lengths of the sides of right angle triangles. We can therefore appreciate how the calculated uncertainty,  $\Delta c$ , is always less than the sum of the added uncertainties (i.e. the hypotenuse is always less than the sum of the sides of a right triangle).



**Figure 4. When uncertainties are added, the resulting uncertainty is calculated by “quadrature.”**

### 5.2 Uncertainty Propagation for Multiplication and Division

Whereas in the case of addition and subtraction, when dealing with uncertainties we dealt with “absolute” uncertainties, in the case of multiplication and division, we need to deal in “fractional” uncertainties.

Fractional uncertainties are expressed in the form  $\Delta a/|a|$ , where  $|a|$  is the “absolute value” of  $a$ .

We follow a similar approach to our derivation of the propagation of uncertainty in the case of addition. Consider the quotient

$$(x \pm \Delta x)(y \pm \Delta y) = (z \pm \Delta z).$$

As before, we consider the uncertainty extremes, where  $x$  is given by  $x+\Delta x$  and  $y$  is given by  $y+\Delta y$ .

$$\begin{aligned} (x + \Delta x)(y + \Delta y) &= xy \left( 1 + \frac{\Delta x}{|x|} \right) \left( 1 + \frac{\Delta y}{|y|} \right) \\ &= xy \left( 1 + \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|} + \frac{\Delta x \Delta y}{|x||y|} \right) \\ &\sim xy \left( 1 + \left( \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|} \right) \right) \end{aligned}$$

(Since  $\Delta x$  and  $\Delta y$  should be small,  $\Delta x \Delta y \sim 0$ .)

Similarly, considering the limiting case of  $x-\Delta x$  and  $y-\Delta y$ ,

$$\begin{aligned} (x - \Delta x)(y - \Delta y) &= xy \left( 1 - \frac{\Delta x}{|x|} \right) \left( 1 - \frac{\Delta y}{|y|} \right) \\ &= xy \left( 1 - \frac{\Delta x}{|x|} - \frac{\Delta y}{|y|} + \frac{\Delta x \Delta y}{|x||y|} \right) \\ &\sim xy \left( 1 - \left( \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|} \right) \right) \end{aligned}$$

Combining the two results and substituting  $z$  for  $xy$ , then we conclude that,

$$(x \pm \Delta x)(y \pm \Delta y) = z \left( 1 \pm \left( \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|} \right) \right)$$

such that

$$\frac{\Delta z}{|z|} = \left( \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|} \right).$$

Finally, we mention that by using the extremes of our uncertainty range (i.e.  $x+\Delta x$ ,  $x-\Delta x$ ) in our derivation, the formula given above again “slightly” overestimates  $\Delta z$ . A slightly more precise version is given by

$$\frac{\Delta z}{z} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

Using a similar derivation, the same result is also found to apply in the case of division.

### 5.3 Summary of Propagation of Uncertainty

Addition and Subtraction:

For

$$(a \pm \Delta a) + (b \pm \Delta b) = (c \pm \Delta c) \quad \text{or} \quad (a \pm \Delta a) - (b \pm \Delta b) = (c \pm \Delta c),$$

$\Delta c$  is given by

$$\Delta c = \sqrt{\Delta a^2 + \Delta b^2}.$$

Multiplication and Division:

For

$$(x \pm \Delta x)(y \pm \Delta y) = (z \pm \Delta z) \quad \text{or} \quad \frac{(x \pm \Delta x)}{(y \pm \Delta y)} = z \pm \Delta z,$$

$\Delta z$  is given by

$$\frac{\Delta z}{z} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}.$$

We now have the important tools required to navigate uncertainty propagation in most scenarios we'll encounter in our experiments.

Before we proceed though we should take a moment to review how we carry significant figures through our calculations.

## 6. Significant Figures and Calculation Methods Involving Uncertainties

### 6.1 Significant Figures

Much as when we described the importance of correctly reading your measuring tool, researchers must also be conscientious in correctly carrying significant figures through calculations performed from their measurements. Several standardized conventions apply.

**Definition:** In any given number, the significant figures include all digits that are considered reliable, as well as the first digit considered doubtful (i.e. the digit defining our uncertainty). Furthermore, we ignore zeros acting as placeholders.

*Examples – Determining Significant Figures*

480491 kg	→	Six significant figures
480500 kg	→	Four significant figures (last two zeros are placeholders)
400000 kg	→	One significant figure (last five zeros considered placeholders)
480.491 g	→	Six significant figures
480.500 g	→	Six significant figures (including last two zero decimals implies they are considered significant)
0.0003489 kg	→	Four significant figures (all zeros before 3 are placeholders)
0.00034 kg	→	Two significant figures
0.034 kg	→	Two significant figures
1.23 g	→	Three significant figures
0.12 g	→	Two significant figures (leading zero in this case is a placeholder, included by convention)
0.120 g	→	Three significant figures (including the final zero in this case implies that it is considered significant)

### 6.2 Calculations, Taking into Consideration Significant Digits and Uncertainty Propagation

It is particularly important to properly account for significant figures when performing calculations, so that results are not stated with too great or too little precision.

#### **Rule 1: Significant Figures when Adding or Subtracting**

When adding or subtracting numbers, the position of the least significant digit in the result is the same as the position of the last significant digit in the least precise number being added or subtracted.

*Simple Examples:*

$$\begin{array}{r}
 4.98\bar{3} \\
 +3.58\bar{7} \\
 \hline
 8.57\bar{0}
 \end{array}
 \qquad
 \begin{array}{r}
 1.287\bar{3} \\
 -0.48\bar{4} \\
 \hline
 0.80\bar{3}
 \end{array}
 \qquad
 3.1415\bar{9} - 2.71\bar{8} + 42.0\bar{0} = 42.4\bar{4}$$

*Experimental Example:*

Say we want to add the masses  $4.51 \pm 0.04$  kg,  $575.62 \pm 0.09$  g, and  $2.1 \pm 0.5$  kg. First, convert to similar units, and then proceed with the calculation as follows,

$$\begin{array}{r}
 4.\underline{51} \\
 0.5756\underline{2} \\
 +2.\underline{1} \\
 \hline
 7.18562
 \end{array}$$

$$\begin{aligned}
 \Delta m &= \sqrt{(\Delta m_1)^2 + (\Delta m_2)^2 + (\Delta m_3)^2} \\
 &= \sqrt{(0.04)^2 + (0.00009)^2 + (0.5)^2} \\
 &= \sqrt{(1.6 \cdot 10^{-3}) + (8.1 \cdot 10^{-9}) + (0.25)} \\
 &= \sqrt{0.2516000081} \\
 &= 0.501 \\
 &= 0.5
 \end{aligned}$$

The last significant digit in each value in the summation is indicated by the underscore. We see that since 2.1 kg is the least precise value in our calculation, its least significant figure determines the overall number of significant figures in our final result. As such, we round up and report that  $m = 7.2 \pm 0.5$  kg.

### Rule 2: Significant Figures when Multiplying and Dividing

When multiplying or dividing a set of numbers, the final result should include only as many significant digits as the number in the set with the fewest significant digits.

Simple examples:

$$1.25 \times \underline{2.0} = \underline{2.5} \quad (\text{Not } 2.50!)$$

$$\frac{4.882}{\underline{2.0}} = \underline{2.441} = 2.4 \quad (\text{Round off to two significant figures.})$$

$$5.286 \cdot \frac{\underline{1.72}}{1.00002} = \underline{9.091738...} = 9.09 \quad (\text{Round off to three significant figures.})$$

Experimental Example:

Say we intend to divide force  $F = 8.24 \pm 0.05$  N by mass  $m = 2.251 \pm 0.006$  kg in order to determine acceleration in accordance with Newton's Second Law, where  $a = F / m$ . Then

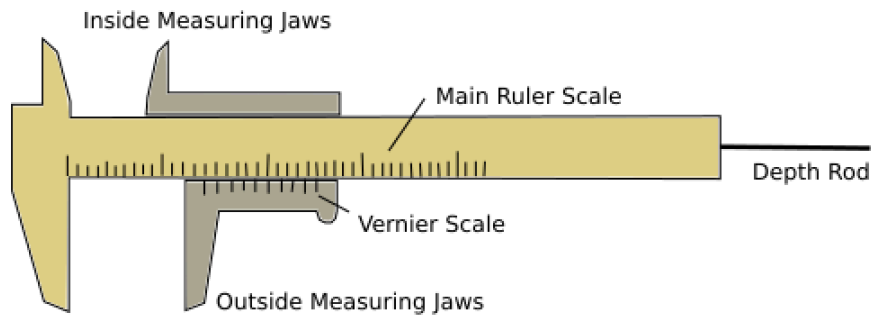
$$\begin{aligned}
 a &= \frac{F}{m} \\
 &= \frac{\underline{8.24}}{\underline{2.251}} \\
 &= 3.66\underline{05}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\Delta a}{a} &= \sqrt{\left(\frac{\Delta F}{F}\right)^2 + \left(\frac{\Delta m}{m}\right)^2} \\
 &= \sqrt{\left(\frac{0.05}{8.24}\right)^2 + \left(\frac{0.006}{2.251}\right)^2} \\
 &= \sqrt{(3.68 \cdot 10^{-5}) + (7.10 \cdot 10^{-6})} \\
 &= 0.0066 \\
 \Delta a &= a(0.0066) \\
 &= 3.6605(0.0066) \\
 &= 0.024
 \end{aligned}$$



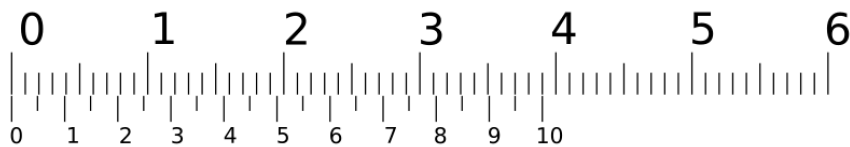
## Appendix I – Reading a Vernier Caliper:

Vernier calipers are a clever modification on a ruler that allows for higher precision readings than the ruler's scale by itself can provide. They are particularly useful for measuring the dimensions of precision machined components.



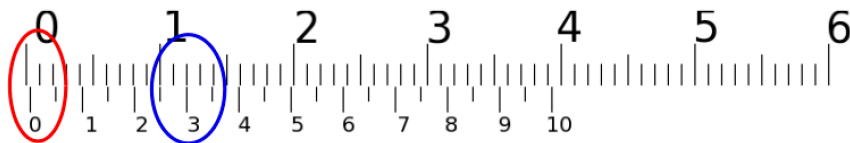
By design, as the jaws of the calipers slide over a distance of 1 mm, the "Vernier scale's" markings align sequentially with the ruler markings above, such that if the jaws are at 0.2 mm then the 2<sup>nd</sup> Vernier scale marking will be aligned with the ruler marking above, and if the jaws are 0.5 mm the 5<sup>th</sup> Vernier marking will be aligned with the ruler marking above, etc.

We demonstrate with a series of examples:



(Note that the zeros on the ruler and the Vernier scale align, and also that the 10<sup>th</sup> marking on the Vernier scale corresponds to the 39<sup>th</sup> mm on the ruler. This "scaled" grating results in sequential shift in marking alignment as the jaws are slid open or closed.)

As we slide the Vernier scale (say we're measuring the diameter of a fibre optic cable) we might have the following:

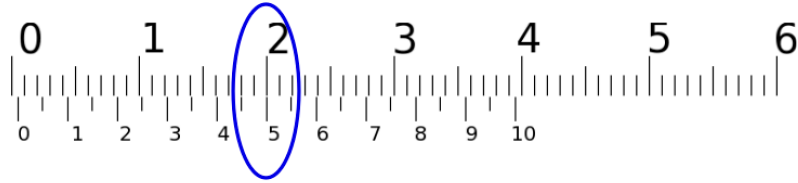


Notice that it would be quite difficult to read the exact distance the Vernier scale's zero has shifted right of the ruler's zero position by eye alone. We could estimate the reading to be somewhere around 0.1 mm to perhaps 0.4 mm. Reasonable, but we can do better...

As we clamp the fibre in our caliper jaw, we see that the Vernier scale actually slid along the ruler such that its "3" aligns nicely with a marking on the ruler. None of the other Vernier scale markings align very well with the ruler markings above. As such, this tells us that the fibre diameter is 0.30 mm (that is, the zero on the vernier scale has moved 3/10<sup>ths</sup> of 1 mm from the zero on the ruler scale).

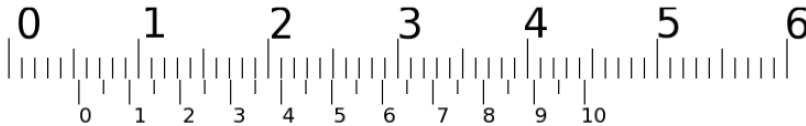
And since we assign uncertainty of half of the smallest division, which is 0.05 mm on our Vernier scale, the uncertainty is  $\pm 0.03$  mm (rounded up from 0.025). So the measurement of the fibre diameter is therefore  $0.30 \pm 0.03$  mm.

Try another...



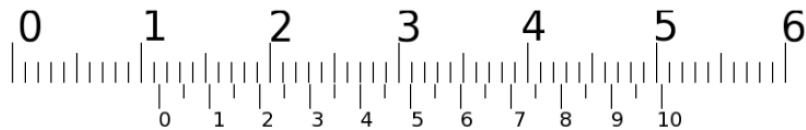
Again, you can see that it would be difficult to estimate directly by eye that the Vernier scale's 0 is 0.50 mm from the ruler's 0 mm marking, but we see quite clearly that the 5 on the Vernier scale aligns with a marking on the ruler above (whereas the other Vernier markings aren't well aligned with the ruler markings), so the reading is  $0.50 \pm 0.03$  mm.

Another example...



The reading would be  $5.40 \pm 0.03$  mm (perhaps even  $5.37 \pm 0.03$  mm, since it's a little hard to say whether 3.5 or 4 align better with the ruler markings above).

Try some on your own...



Solution: \_\_\_\_  $\pm$  \_\_\_\_ mm



Solution: \_\_\_\_  $\pm$  \_\_\_\_ mm